2008 BLUE MOP, FUNCTIONAL EQUATIONS-I
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(1) Find all $f : \mathbb{R} \to \mathbb{R}$ such that $f(f(x)) = x^2 - 2$ for all real x.

(2) (SL-92) Let $a, b > 0$. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy
$$f(f(x)) + af(x) = b(a+b)x, \quad \text{for all } x \in \mathbb{R}^+.$$

(3) (Vietnam-03) Let F be the set of all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ which satisfy the inequality $f(3x) \geq f(f(2x)) + x$ for all positive x. Find the largest positive number α such that for all functions $f \in F$, we have $f(x) \geq \alpha x$.

(4) Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy
$$f(f(n)) + f(n+1) = n + 2.$$

(5) (Belarus-97) Find all functions $g : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$
$$g(x+y) + g(x)g(y) = g(xy) + g(x) + g(y)$$

(6) (BMO-97) Solve the functional equation
$$f(xf(x) + f(y)) = y + f(x)^2, \ \forall x, y \in \mathbb{R}.$$

(7) (SL-03) Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying the following conditions:

(i) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$

(ii) $f(x) < f(y)$ for all $1 \leq x < y$

(8) Given a positive integer n, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(0) = 0$, $f(1) = 1$, and $f^{(n)}(x) = x$ for every $x \in [0, 1]$. Prove that $f(x) = x$ for all $x \in [0, 1]$.

Problem 1, Solution by Brian Hamrick: Let $f^{(k)}$ be f applied k times. $f^{(4)}(x) = x$ is quartic equation with four real roots:

$$a = -1,\ b = 2,\ m = \frac{-1+\sqrt{5}}{2},\ \text{and } n = \frac{-1-\sqrt{5}}{2}.$$

So a, b, m, n are the fixed points of $f^{(4)}$. Moreover a, b are also the fixed points of $f^{(2)}$. Now observe that if x is a fixed point of $f^{(k)}$, then so is $f(x)$. Observe that $f(m)$ and $f(n)$ are m or n because they are fixed points of $f^{(4)}$ but not $f^{(2)}$. On the other hand, f is injective on the set of these four points $\{a, b, m, n\}$ since neither one is the negative of another. Now, in both cases we have $f(f(m)) = m$ which is a contradiction since the only fixed points of $f^{(2)}$ are a and b $\square$

Problem 2, Solution by Wenyu Cao: Let $x_0 = x$, for a fixed x and let $x_{n+1} = f(x_n)$ for $n \geq 0$. The given condition becomes:

$$x_{n+2} + ax_{n+1} = b(a+b)x_n$$

which has the characteristic equation $y^2 + ay - b(a+b) = 0$ with roots $y = b$, and $y = -a-b$. Thus $x_n = sb^n + t(-a-b)^n$ for some real constants s and t. If $t \neq 0$, then for sufficiently large n, the $t(-a-b)^n$ term will dominate the sb^n term and x_n will become negative, which contradicts the definition of f. Thus, $t = 0$ and it follows that $f(x) = bx$ $\square$

Problem 3, Solution by Toan Phan: Firstly, observe that $f(x) = \frac{x}{2}$ satisfies the condition $f(3x) \geq f(f(2x)) + x$. Thus, $\alpha \leq \frac{1}{2}$. Secondly, $f(x) > \frac{x}{3}$. Moreover if $f(x) \geq \alpha x$, then using the functional equation we get $f(x) \geq \frac{2\alpha^2+1}{3}x$, as well. Let $a_1 = \frac{1}{3}$ and $a_{n+1} = \frac{2a_n^2+1}{3}$. Then $\alpha = a_n$ satisfy the inequality for all n. Observe that the sequence $\{a_n\}$ is increasing and bounded by $\frac{1}{2}$, thus it has a limit and we find that $\lim_{n\to\infty} a_n = \frac{1}{2}$. Hence the answer is $\alpha = \frac{1}{2}$ $\square$

Problem 4, Solution by John Berman: Plugging-in values $n = 1$ and $n = 2$, after eliminating some possibilities we conclude that $f(f(1)) = 1$ and $f(2) = 2$. Then by induction we show that for $n \geq 2$, $2 \leq f(n) \leq n$. So the recursion, $f(n+1) = n + 2 - f(f(n))$ determines all $f(n)$ values for $n \geq 2$ uniquely. Then $f(f(1)) = 1$ implies that $f(1)$ cannot be larger than 1, hence $f(1) = 1$. Let $\phi = \frac{\sqrt{5}-1}{2}$. We claim that $f(n) = [\phi n] + 1$. To prove this, we need to show that it satisfies the same recurrence relation as f. Note that $\phi n < f(n) < \phi n + 1$. Furthermore, $\phi k + 1 - \phi < f(k)$ is true for at least one of $k = n$ or $k = n+1$ and $f(k) < \phi k + 1 - \phi$ is true for at least one of $k = n$ or $k = n+1$. Thus,

$$\phi^2 n + \phi(n+1) + 1 - \phi < f(n) < (\phi(\phi n + 1) + 1) + (\phi(n+1) + 1) - \phi.$$

Bounding by integers, $n + 1 < f(f(n)) + f(n+1) < n + 2 + \phi$ implies $f(f(n)) + f(n+1) = n + 2$, as desired $\square$

Problem 5, Solution by Justin Brereton: Observe that $g(0) = 0$ or 2. If $g(0) = 2$ the only solution is $g \equiv 2$. If $g(0) = 0$, letting $x = y = 2$ gives $g(2) = 0$ or 2. Assume that $g(2) = 2$. Then we get $g(1) = 1$ or 2 by letting $x = y = 1$. Assume $g(1) = 1$, then note that $g(x+1) = g(x) + 1$. for all x. In this case, we get $g(x+n) = g(x) + n$ and $g(nx) = ng(x)$ for all x and integer n. Letting $x = \frac{y}{y-1}$, we get $g(\frac{1}{z}) = \frac{1}{g(z)}$ for all non-zero z. Then since $g(n) = n$ for all integers, we get $g\left(\frac{1}{n}\right) = \frac{1}{n}$ and then it follows that $g(r) = r$ for all rational numbers r. Now, since $g(x^2) = g(x)^2 \geq 0$. Combining this with the g fixing rational numbers, we can show that $g(x) = x$ for all real number x. In all the case $g(1) = g(2) = 2$, we get $g \equiv 2$ by letting $y = 1$ which contradicts the fact that $g(0) = 0$. The only case left is $g(2) = 0$. In this case, first show that all rational numbers are sent to zero. Then by letting $y = n$ a positive integer show that $g(x+n) = g(nx) + g(x)$. In particular, $g(x+1) = 2g(x)$, hence by induction $g(x+n) = 2^n g(x)$. So $g(nx) = (2^n - 1)g(x)$. But then, on one hand $g(4x) = (2^4 - 1)g(x)$, on the other hand $g(4x) = (2^2-1)g(2x) = (2^2-1)(2^2-1)g(x)$. We conclude that $g \equiv 0$ in this case as well. In conclusion the only solutions are $g(x) = x$ and the two constant functions $g \equiv 0$ and $g \equiv 2$ $\square$

Problem 6, Solution by Sergei Bernstein: Plugging-in $x = 0$ gives $f(f(y)) = y$. Plugging-in $y = 0$ gives $f(xf(x)) = f(x)^2$. Replacing x with $f(x)$ we get $f(x) = \pm x$. Suppose that $x, y \neq 0$ and $f(x) = x$ but $f(y) = -y$. Then plugging-in (x, y) gives $f(x^2 - y) = x^2 + y$ implies $x = 0$ or $y = 0$, a contradiction. We conclude that $f(x) = x$ and $f(x) = -x$ are the only solutions $\square$

Problem 7, Solution by Sam Keller: Letting $x = y = z = 1$, we get $f(1) = 2$. Then $(x, y, z) = (a^2, 1, 1)$ gives $f(a)^2 = f(a)^2 + 2$. The triple $(x, x, \frac{1}{x})$ then gives $f(x) = f(\frac{1}{x})$. Letting $z = \frac{1}{y}$ and $s = \sqrt{xy}$, $t = \sqrt{\frac{x}{y}}$ gives $f(st) + f(\frac{s}{t}) = f(s)f(t)$. f is increasing for $x = 1$, $f(\frac{1}{x}) = f(x)$ and $f(1) = 2$. We conclude that $f(x) \geq 2$ for all $x > 0$. Let $f(x) = g(x) + \frac{1}{g(x)}$. Inducting on n and using the above equation with $s = x^n$ and $t = x$ we show that $g(x^n) = g(x)^n$. It then follows that $g(x^r) = g(x)^r$ for all rational numbers r. Since g is increasing, the same result follows for all real numbers r and hence by letting $h(x) = \ln(g(e^x))$ and using Cauchy's Equation, we deduce that $g(x) = x^c$ for some constant c. In conclusion $f(x) = x^c + \frac{1}{x^c}$ for some constant c and all such functions actually work $\square$

Problem 8, Solution by Sergei Bernstein: Observe that f is injective. Since it is also continuous and $f(0) < f(1)$, it has to be strictly increasing, otherwise using Intermediate Theorem we would get to different numbers whose images are same. Now if $x \leq f(x)$ applying f to both sides repeatedly and using monotonicity, we get

$$x \leq f(x) \leq f^{(2)}(x) \leq ... \leq f^{(n)}(x) = x.$$

Hence $f(x) = x$. We get the same result in the other case: $x \geq f(x)$ and conclude that $f(x) = x$ for all $x \in [0, 1]$ $\square$